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# KIND OF WARRANTS AND PROOFS USED BY UPPER SECONDARY STUDENTS IN INTUITIVELY APPROACHING THE DENSITY OF $\mathbb{Q}$ IN $\mathbb{R}$

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*While research in Mathematics Education has mainly focused on the density of  $\mathbb{Q}$ , little is known about the conceptualization of the density of  $\mathbb{Q}$  in  $\mathbb{R}$ . This study investigates upper-secondary school students' warrants and proofs regarding the density of  $\mathbb{Q}$  in  $\mathbb{R}$ . A classroom activity focused on continued fractions, which provide a unified representation of rational and irrational numbers, was carried out. Through warrant- and proof-analysis, this study reveals how different kinds of warrant and proof shape the students' reasoning on the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , providing insight into the level of mathematical depth reached and the reasoning effectiveness produced by students.*

## RATIONALE

The concept of density of  $\mathbb{Q}$  is a well-known obstacle in the transition from  $\mathbb{N}$  to  $\mathbb{Q}$ , and much research has focused on it (e.g., Hannula et al., 2012; Kim & Kwon, 2019; McMullen & Van Hoof, 2020), whereas the density of  $\mathbb{Q}$  in  $\mathbb{R}$ —i.e., the property that given two real numbers, there is always a rational number between them—, a key concept related to the transition from  $\mathbb{Q}$  to  $\mathbb{R}$ , has received little attention from research in Mathematics Education (Marmur et al., 2020). The transition from  $\mathbb{Q}$  to  $\mathbb{R}$  becomes crucial during the calculus courses at the university level when  $\mathbb{R}$  must be conceptualized as the domain necessary to investigate continuous phenomena and limits. Indeed, Marmur et al. (2020) investigate how first-year mathematics undergraduates grasp the concept of density of  $\mathbb{Q}$  in  $\mathbb{R}$ . Actually, students intuitively approach this concept already in upper-secondary school and acquire spontaneous conceptions—in the sense of intuitive beliefs based on experiences in various contexts, including everyday life (Fujii, 2014)—of it, on which the future institutionalized concepts will build. However, spontaneous conceptions are not directly accessible, and the individual is usually unaware of them. Investigating the type of warrants, i.e., the reasons expressed in an inferential step of an argument (Toulmin, 1958), and proofs—in the sense of Balacheff (1988)—that students use when they reason on the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , or on concepts related to it, can be useful to understand how students spontaneously approach this concept. To this end, students' arguments used to support their reasoning in working on a task involving continued fractions are categorized and integrated, focusing on the following research question: What kind of warrants and

proofs do students use in facing a task that implicitly involves the concept of density of  $\mathbb{Q}$  in  $\mathbb{R}$  and how do these warrants and proofs shape their reasoning? The paper presents a part of the results of such warrant- and proofs-classification of data collected in two classes of Italian upper-secondary students (aged 17/18).

## THEORETICAL FRAMEWORK

Toulmin (1958) defines the “skeleton of a pattern for analyzing arguments” (p. 99) as a structure consisting of data (the argument starting point, i.e., the evident facts), conclusion (the assertion that needs to be justified), and warrant (the logical link between data and conclusion). Even if both Toulmin and other authors added further elements to the original skeleton, the basic structure is sufficient for reconstructing the argumentation, i.e., the argumentative process in our context. We explicitly clarify that, in this paper, we use “argumentation” to indicate the overall argumentative process, whereas “argument” is for the single sentence examined during the analysis.

Rodd (2000) distinguishes between warrant, i.e., “that which secures knowledge” (p. 222), and justification, i.e., “rationale for a belief” (p. 222). If a justification warrants the truth of a belief, then it is a warrant, or better, a mathematical warrant; otherwise, it is not, and Rodd calls it simply justification. In our context, we do not make this distinction. Indeed, our research participants are in the initial phase of the intuitive approach with the mathematical object inquired—which still must be formally introduced—and we thus consider as a warrant also students’ arguments that are incomplete, incorrect, or expressed through registers unsuitable for mathematical formalization, e.g., the visual representations. Indeed, according to Rodd (2000), “visualization does not negate or detract from proofs of a deductive–symbolic character” (p. 236), and we accept it as a ground of a mathematical warrant.

We use an amount of seven warrant-categories to account, from an epistemic point of view, for the types of warrants students use during their reasoning. A first distinction was made between *pragmatic warrants* (related to utility and efficiency) and *theoretical warrants* (related to properties and definitions). The warrant-categories defined by Inglis et al. (2007)—*inductive warrant* (based on the use of a certain number of examples), *intuitive warrant* (based on the use of observations or examples that reflect some underlying cognitive structure), and *deductive warrant* (based on deduction from axioms, algebraic manipulations, and counterexamples)—are specific kinds of theoretical warrant. Subsequently, for each of the warrant-categories defined by Inglis et al. (2007), a distinction between *operational warrant* (related to a procedure) and *structural warrant* (related to properties and relationships between objects) was made. The first two distinctions—one between pragmatic and theoretical warrant and one between the three types of theoretical warrant—were made because of the participants’ reduced mathematical experience. Indeed, certain warrants (e.g., theoretical warrants) may rely on competencies (e.g., linguistic and cognitive skills) that students may not yet possess, whereas others (e.g., pragmatic warrants) are more accessible in this context. Instead, the latter distinction (between operational and

structural warrants) is a helpful descriptive tool in the warrant-analysis process, which distinguishes the type of warrant used in the argumentation more in detail. Defining the structural intuitive warrant, Inglis et al. (2007) implicitly refer to Rodd's (2000) idea of "visual arguments" (p. 5). Indeed, this kind of warrant is defined by the authors as "some kind of mental structure, be it *visual* or otherwise, that persuades them [the participants] of a conclusion" (Inglis et al., 2007, p. 11) (*italic* introduced by the authors).

To account, from a logical point of view, for the different kinds of proof that are not necessarily mathematical in Rodd's (2000) sense but are important from an epistemic perspective, we use Balacheff's (1988) classification of proofs. Balacheff distinguishes between *pragmatic proof* (based on actions) and *intellectual proof* (which employs verbalizations of the properties of objects and their relationships). This proof-classification accounts for the level of generality achieved by students.

Additionally, Balacheff identifies a taxonomy for the types of proofs used in Grisendi (2024). However, since we do not refer to these categories in this paper, we will not dwell on these further.

## RESEARCH METHOD

To stimulate the emergence of spontaneous conceptions, an activity focused on continued fractions, which facilitates the transition from  $\mathbb{Q}$  to  $\mathbb{R}$ , was designed. Two classes for a total of 42 upper-secondary school students were involved. The activity, divided into two parts, was conducted under the lead of the first author in the presence of the classroom teachers but without their intervention. In the first part (two hours), students respond anonymously, working in small groups. This part involves a worksheet, which begins with a brief contextualized introduction to continued fractions, where a hypothetical student proposes a continued fraction as an approximation of an irrational solution of a second-grade equation; then, students are asked to argue for or against the statement (called Statement below): "Given any real number, can I always find a rational number that is as close to it as I want?" After reading the initial part of the worksheet, students complete two parts, called Assignment 1 and Assignment 2. In Assignment 1, students write down a points list of arguments for or against the Statement, while in Assignment 2, starting from the generated list, students are asked to write an argumentative text. These responses were used to track students' reasoning processes and identify the kinds of arguments they used to support their conceptions. Finally, students are interviewed by the researcher a few days later via semistructured group interviews, inviting them to deepen the argumentation presented by their group. In this sense, the collected data were written or spoken sentences produced by the students while arguing their position concerning the task claim.

We use qualitative content analysis (Mayring, 2015) and structure the analysis-outputs into three interconnected levels to explore the progression of each group's reasoning. Indeed, our data analysis approach draws inspiration from Toulmin's (1958) simile:

“An argument is like an organism. It has both a gross, anatomical structure and a finer, as-it-were physiological one” (p. 94). Firstly, the “Initial framing”-phase provides a holistic overview of the reasoning, presenting the group’s argumentation “anatomical structure.” Secondly, the “Detailed analysis and classification”-phase analyses each warrant and proof, describing the group’s argumentation “physiological structure.” Here, individual sentences are examined with the classification system explained in the Theoretical framework. Finally, after providing a holistic overview and a granular analysis of the argumentation, we explore connections and transitions between sentences, focusing on how the identified warrant and proof-categories reveal the characteristics of students’ reasoning as they approach the topic. In this sense, the third and final phase, “Concluding discussion”, bridges between the first two phases, synthesizing the findings and offering a global vision of the group’s arguments.

## DATA ANALYSIS AND RESULTS

From the analysis of the twelve groups, only five of the seven warrant-categories defined in the Theoretical framework emerge. We focus only on four groups because these groups—analyzed more in-depth in Grisendi (2024)—cover all the warrant-categories that emerged from the data. For each warrant-category analyzed, we provide a representative example. The analysis of each group starts with the warrant used by the group; then, we present the proofs detected in reconstructing the group’s argumentative process. Presenting the results, we move from the least effective reasoning for conceptualizing the topic to the most effective.

### Group 6 – Operational inductive warrant

**Initial framing.** From the responses provided by Group 6, it can be inferred that the legitimacy of the Statement is perceived as evident and does not require further consideration. The students focused on verifying the validity of the procedure involving continued fractions rather than providing an argumentation to support the proposed claim. **Detailed analysis and classification.** In Assignment 2, the students wrote: “[...] Given any real number, it is possible to obtain an irrational number arbitrarily close to the original. Starting from [the procedure using continued fractions], one can get increasingly closer to one of the results of the exercise. To demonstrate this, a list of different values was tested, including negative, positive, and decimal numbers.” Starting from the statement, “To demonstrate this, a list of different values was tested,” the argument can be classified as an operational inductive warrant because the students supported their position with a series of tested examples to demonstrate the validity of the procedure presented in the task. During the interview, students were asked to clarify what they meant by “negative, positive, and decimal numbers.” This revealed confusion regarding the nature of numerical sets and the relationships between them. **Concluding discussion.** The lack of prerequisites concerning numerical set distinctions and the students’ difficulty, in this context, in investigating the core concept, i.e., the density of  $\mathbb{Q}$  in  $\mathbb{R}$ , is exemplified through an operational inductive warrant.

### Group 5 – Pragmatic warrant and pragmatic proof

**Initial framing.** Group 5 focused primarily on solving the equation without fully grasping the objective of the activity. They paid attention to aspects relevant because of the didactic contract, such as the need to specify an algebraic fraction's existence conditions and the Statement's domain of validity. **Detailed analysis and classification.** In Assignment 2, the students wrote: "The statement [...] is not universally correct because the [procedure with continued fractions] is correct [...] and allows one to get closer and closer to the solution [...], but it does not apply to all real numbers. Indeed, if I use the solutions of the equation, I do not obtain a number close to but equal to the solution." For Group 5, the Statement is legitimate because it is based on a useful procedure, as highlighted by the phrase "[the procedure] allows one to get closer and closer to the solution." In this sense, this argument is classified as a pragmatic warrant. However, the group also argued that the Statement is incorrect because, despite its basic legitimacy, it does not hold for "all real numbers." Because students show an example based on the act of substitution, "If I *use* the solution" (italic introduced by the authors), this argument falls under the category of pragmatic proof. **Concluding discussion.** Using a pragmatic warrant and a pragmatic proof likely indicates students' avoidance of effective reasoning in this context. Nevertheless, the substitution act that characterizes the pragmatic proof enables students to consider the issue of the Statement and approach the research topic.

### Group 2 – Structural intuitive warrant

**Initial framing.** Students from Group 2 effectively framed the problem using a Venn diagram to represent numerical sets. Although this is only implicitly evident in their written work, the graphical tool enabled Group 2 to visualize the relationships between the various sets involved, reflecting an underlying cognitive structure. **Detailed analysis and classification.** The statement from Assignment 2, "Considering that real numbers are divided into rational and irrational numbers, we need to try to demonstrate that every number belonging to these two sets can be expressed as a fraction," is classified as an intuitive warrant because it represents an observation based on a mental structure that the students have constructed as they developed their initial reasoning. Additionally, this statement can be classified as a structural warrant, as the students refer to a "property" of rational and irrational numbers—namely, the property of representing these numbers in fractional form. **Concluding discussion.** The structural intuitive warrant is supported by a visual representation (Rodd, 2000; Inglis et al., 2007), which helped the students clarify their ideas about the relationships between numerical sets and propose a distinction of cases to base their reasoning on. Specifically, the students distinguish between irrational and rational numbers, referring to the distinctive "property" related to fractional notation. In this sense, the students recognize the role of continued fractions in this context, which bridges notation between the two categories of numbers and thus between  $\mathbb{Q}$  and  $\mathbb{R}$ .

## Group 10 – Operational and structural deductive warrant and intellectual proof

**Initial framing.** Group 10 begins with the observation that irrational numbers are characterized by an infinite number of decimal digits. These characteristics form the foundation for constructing an iterative approximation method. **Detailed analysis and classification.** In Assignment 1, the students base their reasoning on the statement: “Given an irrational number, it will have infinite digits after the decimal point.” This represents a “property” of irrational numbers and is categorized as a structural deductive warrant. Subsequently, the students write: “We can truncate [the irrational number] at a point of our choice, but after doing so, we could decide to keep one more digit than before.” This statement outlines a procedure and is therefore classified as an operational deductive warrant. This reasoning leads the students to develop, in Assignment 2, an intellectual proof. The statement classified as an intellectual proof is the following: “We truncate the irrational number at any point, thus obtaining a rational number; however, at this point, we can keep one more digit than before, still resulting in a rational number.” We can observe that, while in the theoretical warrant, it is possible to distinguish between the theoretical concept and the operational procedure, in the intellectual proof, the two components produce a unified and coherent reasoning process. **Concluding discussion.** The warrant- and proof-categories used by Group 10 students demonstrate an original and well-focused approach to the topic under investigation, showing the ability to produce a general argumentation. The reasoning performed by Group 10 enabled the students to conceive the cardinality of the continuum of the set  $\mathbb{R}$  intuitively. This is evident during the interview when, referring to the arrangement of numbers on the real line, one of the students states: “There’s a little value in every spot,” followed by another student affirming: “There’s no space between one number and the next.”

## DISCUSSION AND CONCLUSION

Thanks to the three levels of analysis shown before, we offer a comprehensive view of each group’s argumentative process, providing insight into the level of mathematical depth students reached and the reasoning effectiveness students produced. On one hand, in this context, “mathematical depth” refers to students’ ability to transcend specific examples or surface-level observations, uncovering broader structures, relationships, and implications in a mathematical framework. On the other hand, in this context, reasoning effectiveness indicates the objective—i.e., according to Rodd (2000), the “general, nonnegotiable, and reliable” (p. 242)—of the reasoning. Indeed, Group 6 students’ reasoning is less effective than that of others because it relies on a negotiable case system (“negative, positive, and decimal numbers”), which highlights students’ difficulties in distinguishing numerical sets. For this reason, it is unlikely to be reliable. We could say that there is a positive correlation between mathematical depth and reasoning effectiveness, in the sense that effective reasoning produced by a group may indicate abilities to achieve a higher level of mathematical depth regarding the concept because reaching a higher level of mathematical depth deserves general or generalizable arguments, which in turn are indicators of effective (mathematical)

reasoning. Figure 1 highlights this relationship: as the effectiveness of reasoning increases, so does the mathematical depth level. The schema illustrates the warrant- and proof-categories that emerged from the groups' argumentative processes. The groups are arranged from left to right, moving from the least effective reasoning to the most effective. Each group corresponds to a cell in the figure. The cell's height reflects metaphorically and in a relational way the level of mathematical depth of the group's reasoning—the taller the cell, the higher the level of mathematical depth.

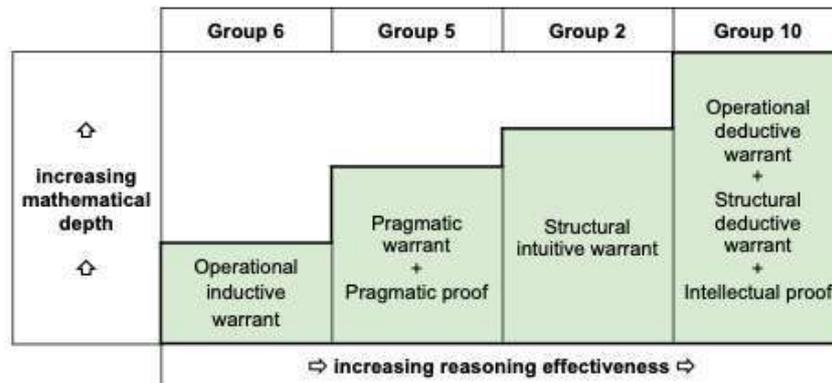


Figure 1: Schema of reasoning effectiveness and mathematical depth across groups

We provide a cross-sectional perspective across the four analyzed groups to address the second part of the research question. From the gaps of Group 6 to the advanced reasoning of Group 10, we highlight the higher and higher level of mathematical depth achieved. Group 6 students remain at a pre-conceptual stage, whereas Group 5 takes a modest step forward by approaching the core concept. However, both rely on practical observations rather than structured reasoning, which indicates only a superficial level of mathematical depth. By visualizing numerical set distinctions, Group 2 demonstrates a more significant conceptual leap, achieving an intermediate level of mathematical depth. Finally, Group 10 attains the highest level of mathematical depth, constructing a general argumentation, which develops into an intuitive conceptualization of the cardinality of  $\mathbb{R}$ .

This analysis shows that the level of mathematical depth and the effectiveness of the reasoning correspond to the types of warrant and proof-categories students rely upon and the interconnectedness of their arguments. Indeed, students whose reasoning demonstrates multiple and mutually reinforcing categories achieve a higher level of mathematical depth and more effective reasoning than those producing isolated and/or “basic”—i.e., not general—arguments.

Further research is needed to deepen the role of different kinds of warrants and their interplay with various types of proofs in conceptualization.

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